1/24/26

Last time:

<u>Ex</u> (*) Is the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ a linear comb. of the vectors $\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}$?

We'll show that answering (*) is the <u>same</u> as solving a certain linear system!

(*) asks: can we find x_1, x_2, x_3 st

$$\begin{bmatrix}1\\1\\3\end{bmatrix}(x_1) + \begin{bmatrix}1\\0\\2\end{bmatrix}(x_2) + \begin{bmatrix}2\\1\\1\end{bmatrix}(x_3) = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

ie Solve

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ x_1 \quad\quad + x_3 &= 2 \\ 3x_1 + 2x_2 + x_3 &= 3 \end{aligned}$$

uniq sol'n $x_1 = 7/4$, $x_2 = -5/4$, $x_3 = 1/4$ (leave it to you to check)

In general:

Say A is an $n \times m$ matrix, so there are m columns.

Say the columns are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$. let $\vec{x} = \begin{bmatrix}x_1\\ \vdots \\ x_m\end{bmatrix}$

then

$$\boxed{\begin{aligned} A\vec{x} &= \underbrace{\begin{bmatrix} \vec{v}_1 & | & \vec{v}_2 & | & \cdots & | & \vec{v}_m \end{bmatrix}}_{A} \begin{bmatrix}x_1\\ \vdots \\ x_m\end{bmatrix} \\ &= x_1\vec{v}_1 + \cdots + x_m\vec{v}_m \end{aligned}}$$

the book stresses these equations!

More on matrix algebra:

__Theorem__

If A is an $n \times m$ matrix and $\vec{x}, \vec{y} \in \mathbb{R}^m$ and k is a scalar then

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} \qquad \text{(distributive property)}$$

and $$A(k\vec{x}) = (kA)\vec{x} = k(A\vec{x})$$

(This will turn out to be an important property, when we start __linear transformations__.)

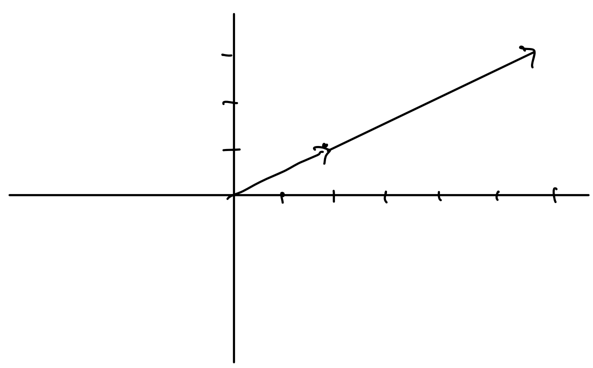
__Ex__ $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 15 \\ 31 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 26 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 15 \\ 31 \end{bmatrix} \quad \checkmark$$

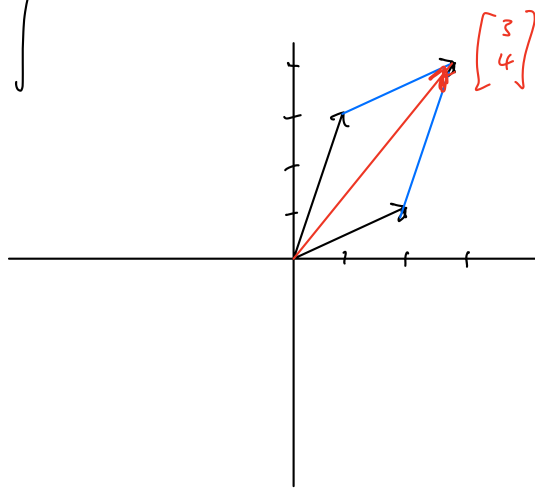
Geometric interpretation in $\mathbb{R}^2$:

- scalar mult. __stretches__ the vector
- vector addition gives a parallelogram.

__Ex__ $$3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$



$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



imagine moving one of the vectors so it starts at the end of the 2nd vector.

See Appendix A, Geometrical Representation of Vectors, for more along this direction.

See Appendix A, Theorem A.2, for standard formulas about vectors.

Now move to Ch. 2, specifically § 2.1

I'll leave it to you to read the motivating example in §2.1 about being in the French Coast Guard radioing your position to Marseille.

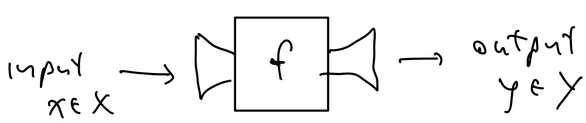
Start just above Figure 3.

A linear transformation is a (very) special kind of function, so let's start there.

Recall:

Def Let X, Y be sets. A function f from X to Y $\boxed{\text{written } f : X \to Y}$ is a rule that associates to each $x \in X$ a unique element $y \in X$. X is the domain of f and Y is the target space.

Think of f as a "black box"



For us, X and Y will be vector spaces $\mathbb{R}^m$, $\mathbb{R}^n$

<u>Def</u> A function $T: \mathbb{R}^m \to \mathbb{R}^n$ is a <u>linear transformation</u> if there exists an $n \times m$ matrix A such that

$$T(\vec{x}) = A\vec{x}$$

for all $x \in \mathbb{R}^m$.

Note: the input is a vector in $\mathbb{R}^m$, the output is a vector in $\mathbb{R}^n$.

<u>Ex</u> $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ 3×2 matrix $n=3,\ m=2$ $T: \mathbb{R}^2 \to \mathbb{R}^3$

Say $\vec{x} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. $T(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \\ 19 \end{bmatrix}$

More generally, if $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $T(\vec{x}) = \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Then

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

so

$$x_1 + 2x_2 = y_1$$
$$3x_1 + 4x_2 = y_2$$
$$5x_1 + 6x_2 = y_3$$

<u>Ex</u> Say $m = n$ and T is the identity function $T(\vec{x}) = \vec{x}$ for all $x \in \mathbb{R}^n$. (i.e. the output is always equal to the input in this example.)

Then $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

ie
$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ &\vdots \\ y_n &= x_n \end{aligned}$$

So $A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ identity matrix

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Above we gave a result that reflects the most important property of a lin. trans.
we said:

Theorem

If A is an $n \times m$ matrix and $\vec{x}, \vec{y} \in \mathbb{R}^m$ and k is a scalar then

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$$

and $$A(k\vec{x}) = (kA)\vec{x} = k(A\vec{x})$$

In our new language:

Cor if $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a lin. trans. and $\vec{v}, \vec{w} \in \mathbb{R}^m$, and $a, b \in \mathbb{R}$

then

$$T(a\vec{v} + b\vec{w}) = aT(\vec{v}) + bT(\vec{w}).$$

Think about why this is true.